TEACHING THROUGH PROBLEM SOLVING: ASSESSING STUDENTS' MATHEMATICAL THINKING

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Abstract: *This paper describes how assessment of students' mathematical thinking was integrated to classroom teaching by using problem solving to introduce a concept that was new to them.*

Towards Transforming Classroom Teaching, Learning, and Assessment

A New Experience for Teachers

The lessons that are described in this paper were collaboratively developed by the four Mathematics IV (Grade 10) teachers of Sta. Lucia High School and the author. The teachers participated in lesson study for the first time. Observation of their classes by the author before they got involved in lesson study revealed that the emphasis of their teaching was for students to learn facts and procedures and apply these in routine problem solving. Their teaching strategy was mostly exposition which often did not build on students' previous knowledge and skills. Likewise the focus of their assessment which was mainly through writing was to test if students mastered these facts and procedures and could use them to solve routine problems.

As part of the orientation-workshop on lesson study for the teachers, they participated as "students" in a lesson that introduced quadratic equation through problem solving. The lesson involved multiple representations and multiple solutions. Through it, the teachers experienced what it meant to engage in mathematical thinking while learning mathematical content. And they became open to the possibility of teaching this way, which was different from how they had been teaching. But in order for them to provide mathematical thinking experiences to their students, the teachers first needed to experience how to collaboratively develop and teach their own lessons that would elicit students' mathematical thinking. If they could provide opportunities for their students to think mathematically as they learned mathematics, then they could gather evidences on how they think mathematically and could better support their learning.

So in their first lesson study, the teachers planned to teach through problem solving which made it possible for them to integrate assessment to their classroom teaching. Evidences that they could gather on how students think, what they know, what they can do, and what their dispositions are would be used as important basis for instructional decisions to further improve students' mathematical thinking and learning. Since mathematical thinking is complex, multiple sources of information regarding it should be used (Webb 1993). Observations and students' practical work outputs, written work, and their group work presentations and discussions, would be used to gather information regarding their mathematical thinking during the lessons.

Planning a New Experience for Students

Problem solving should not only be a product of learning mathematics but should also be a medium for learning it (Webb 1993). So the teachers decided that they would teach polynomial function, a concept that the students would encounter for the first time, through problem solving. They believed that when students solved a problem to learn a concept that was new to them, they could demonstrate their present learning status. Before this topic, the students had already studied relations, functions, linear function, and quadratic function. The lessons on polynomial function intended to provide them plenty of experiences to reason, represent situations mathematically, and relate their previous knowledge to what they would presently learn.

The way the lessons would be implemented is now described. In groups, the students would be asked to devise their own procedures of making the desired boxes. This would be the problem. From the process and product of solving the problem, they should discover that there are changing quantities and functional relationships which they would be asked to represent mathematically. One of these quantities is the volume of the boxes. Its representation would be used to introduce the concept of polynomial function based on what students already know about polynomials and functions.

Specifically, the objectives of the lessons that are considered in this paper are: (1) to construct boxes of different sizes with an open top, (2) to represent real-life situations mathematically and give meaning to those representations, and (3) to relate polynomials and functions to polynomial function. The lessons had three parts. The first was making the different boxes. The second was gathering and presenting data that could be obtained from the boxes. The third was representing the relationships shown by these data and using these representations to develop the concept of polynomial function.

The class that is described in this paper is Section 7 out of 15 sections in Mathematics IV. It consisted mostly of average to low ability students.

The problem

Christmas is fast approaching. Lucy wants to give a personalized gift to her friends. She plans to make a box with an open top where she can store the gifts. Before making the actual box, she wants to try it first by using a plain sheet of grid paper measuring 10 cm by 16 cm. If you were Lucy, what possible boxes can you make? Among the boxes that you made, which do you prefer and why?

Pointers to follow:

1. Construct an open-top box of different sizes using a sheet of grid paper, pair of scissors, and tape.
2. Use one sheet of grid paper for each box.
3. Do not remove any part of the sheet of grid paper.
4. Avoid any folds on the top part of the box.
5. Do not waste the sheets of grid paper so that you can make many boxes.

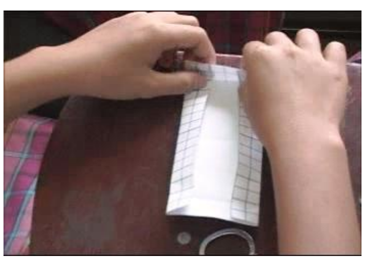
For item 3, the teacher said that the students could cut the paper but not cut off any part.

Integrating Assessment to Classroom Teaching

Visualizing Relationships in Box-making

Solving problems by working out plans then evaluating them and improving the process and the results is part of the mathematical activities in inquiry-oriented classrooms (Shimizu 2010). Observations of the implementation of the lessons provided rich data on how the students came up with their own procedures in making the boxes. They planned the procedures, carried them out and evaluated if these gave the required results. If they did not, they made adjustments on their procedures until they were able to make the desired the boxes.

Apparently, the students visualized relationships in making the required boxes. It could be inferred that they thought that a box with an open top should have a base. It should also have four other faces. These could be formed by folding along the four sides of the sheet of grid paper. They folded along the sides such that the rectangles formed have the same width to ensure that there would be no folds at the top parts. The folds created a square at each corner of the sheet of grid paper. They then cut one side of each square to have a flap at each corner. The procedures described are shown on the pictures below.



These are some interesting findings. There was a group of students which just folded at the corners and did not cut so they had triangular instead of square flaps. There were groups that folded at the top part of one pair of opposite faces of a box because when they folded along the four sides of the sheet of grid paper, the widths of the rectangles formed were not the same.

Most of the groups were able to make the required boxes. And of these, except for two groups, they made four boxes whose dimensions were all integral values. They got these results by folding along the grid lines of the sheet of grid paper. Two groups were able to explore other possibilities. One of them made a box whose height was 2 ½ cm; the other, with a height of 4 ½ cm.

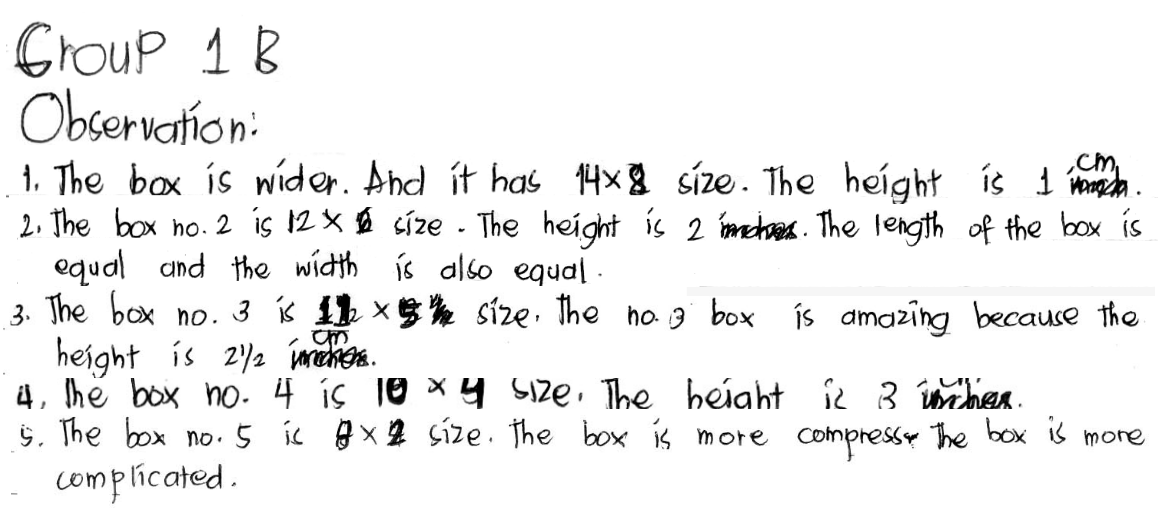
The students had different box preferences. Supposedly, the box should hold gifts temporarily. The box with the height of 1 cm was chosen because it was long and wide; with the height of 2 cm, because it was easier to make; and with the height of 3 cm, because it had more space. No one chose the box with the height of 4 cm because it was difficult to make. So the choices were based not only on the purpose of the box but also on how easily it was made.

Gathering and Organizing Data on Changing Quantities

The teacher asked the students to write all the data that they could about their boxes. This activity involves mathematizing and utilization of information. These are also parts of the mathematical activities in classes that use the inquiry approach (Shimizu 2010). Mathematizing includes observation and examination focusing on number, quantity or shape of things and grasping exactly the properties of things. Utilization of information includes arranging, classifying, and choosing the necessary information for making decisions.

It was solely the decision of the students what data they would gather, how they would organize and present them, and what sense they could make out of those data. Just like in the box-making activity, the responses of the groups of students in this activity revealed the extent and depth of their mathematical understanding and the quality of their mathematical thinking. The works of only four groups are presented here.

No group referred to the boxes in terms of their dimensions to distinguish them from one another. Rather, they labelled them as Box 1, Box 2, and so on based on increasing heights. Group 1B was close to naming their boxes based on their dimensions as shown below.


Group 1 B
Observation:
1. The box is wider. And it has 14×8 size. The height is 1 ~~inch~~ cm.
2. The box no. 2 is 12×6 size. The height is 2 ~~inches~~. The length of the box is equal and the width is also equal.
3. The box no. 3 is 11 × 5 size. The no. 3 box is amazing because the height is 2½ ~~inches~~ cm.
4. The box no. 4 is 10 × 4 size. The height is 3 ~~inches~~ cm.
5. The box no. 5 is 8 × 2 size. The box is more compress. The box is more complicated.

There were groups such as 1B (whose work was shown earlier) and 3A (whose work is shown below), that presented both qualitative and quantitative information about their boxes. The former were based on the appearance of the boxes; the latter, on their measurements. But they did not consistently qualitatively and quantitatively describe every box. Their descriptions involved comparisons with other boxes.

GROUP "3-A"

box #1

1) If the height of the box is one centimeter, the wider it can occupy more gifts.

box #2

2) The box with the height of two centimeters has the lesser space to be occupied than the biggest box.

box #3

3) The taller the box, the lesser space It gives.

box #4

4) It is the tallest box but ~~the~~ its space ~~has~~ is the least of all/among the rest.

Group 3A which was referred to above and Groups 5A and 4A whose works are shown below, observed that the height of a box was related to the amount of space it had. In fact, Group 4A generalized that the smaller the height, the more space it had or the bigger the height, the smaller space it had. No group referred to the "amount of space" as volume. Even Group 4A which determined the length, width, and height of the boxes, did not compute the volume but computed the area and perimeter of a face that they did not identify. Apparently, its basis for comparing the amount of space of the boxes was solely the appearance of the boxes.

Group 4-A

Observation:

The boxes that we made have different sizes. The box that has a lower side, has more space to occupie And the box that has a deeper side, has a less space to occupie.

box #01:	box #2	box #3	Box #04
size:	size	size	size
Lenght: 14 CM	lenght: 12 CM	lenght: 10 CM	lenght: 8CM
widtH: 8 CM	widtH: 6 CM	widHt: 4 CM	widtH: 2CM
~~Height~~ perimeter: 44CM	Height: 2CM	Height: 3 CM	Height: 4CM
Height: 1CM	Perimeter: 36 CM	perimeter: 28CM	Perimeter: 20 CM
Area: 112 CM^2	Area: 72 CM^2	Area: 40 CM^2	Area: 16 CM^2

Notably, Group 5A whose work is only partially shown here computed the volume of the boxes but labelled it as area. It seemed that the members generalized that the smaller the height of a box, the bigger its amount of space. Apparently, they did not use their computed values of the product of the length, width, and height to verify what they thought might be true about the amount of space of a box relative to that of the other boxes based on what they saw. Possibly, they did not associate "amount of space" to this product. Or, if they did although they mislabelled it, they did not bother to verify if their observation was correct by referring to their computations.

Group 5/A

Observation in the Boxes:

1) In Box no. 1 the space is morethan space to other box and it has a height of 1 cm on box no. 1

- formula: Area = LWH

  Area = (14)(8)(1)

  $\boxed{\text{Area} = 112}$

2.) The box no. 2 has smaller than no. 1 but the height is greather than the Box no. 1 and the width of the box. 2 is less than in Box. 1.

- formula : Area = LWH

  Area = (12)(4)(2)

  $\boxed{\text{Area} = 144}$

Not shown in the works presented here but was captured in the video was a girl who, looking at their tabulated data, told her group mates that the values of the length and the width of the boxes were decreasing by two. The group noted that this happened as the values of the height increased by 1. Recognizing this pattern could have been the start of realizing that the length and the width of a box are functions of its height.

There are notable inferences that can be formulated based on the above findings related to what sense the groups made out of the data that they gathered. One is that the students did not have a clear understanding of the concept of volume and area both in terms of their meaning and formula. Another is that they did not realize that the generalization that they made based only on their observations might not be right and that they should use their computations to verify. In short, the generalization that they formulated could not be deduced from the values that they computed based on the data that they gathered. Still another is that the students made a generalization based on what could easily be observed such as the amount of space of a box in relation to its height. However, they could not do so when the relationship was not very visually striking such as that between the length and width of a box and its height.

Representing Relationships

In the previous activity, the students visually noted a relationship between the height of a box and the amount of space it had. This time, they were to discover relationships based on the data that they gathered and represent these relationships mathematically. Mathematical interpretation of phenomena and mathematical representation of solutions are also parts of inquiry-oriented mathematical activities (Shimizu 2010).

The table of values of length, width, and height shown below was written on the board by a group.

Box #	Length	Width	Height
1	14 cm	8 cm	1 cm
2	12 cm	6 cm	2 cm
3	11 cm	5 cm	2 ½ cm
4	10 cm	4 cm	3 cm
5	8 cm	2 cm	4 cm

The teacher wanted to know if the students could explain how the values were obtained. That is, if they could represent the process mathematically to show the relationships among the changing quantities. To accomplish this, she asked a series of questions. Specifically in the following transcriptions, she tried to determine if the students could relate the height of a box to the length of a side of the squares that served as flaps at the corners of that box.

Speaker	Transcriptions
Teacher	This height... Where did it come from the plain sheet of paper? [points to the height of the 14 x 8 x 1 cm^3 box]. What is this?
Student	Width
Teacher	Before this became a box ... How did you make your box?
Student	Fold.
Teacher	Where? What's the first step did you do?
Student	Cut the corners.
Teacher	What figure was cut or what figure did you fold?
Students	Triangle/Rectangle/Square
Teacher	The figure is a square. What is the side of the square that has been cut if this is the box?

Students	One
Teacher	The length of the side of the square is one. After cutting it, you folded. After folding the flaps, what happened? What is the relation of the length of the side of the square to the box?
Students	Height
Teacher	It became the height of the box. You cut one centimetre on a side of the square. When you folded, it becomes the height of the box. It is this. So one centimetre (points to 1 under the column height on the board).

So the students knew how the values of the height were obtained. Next, the teacher tried to determine if they knew how the values of the length and the width of each box were obtained in relation to its height. Each time for a specific box, she would point to the opposite ends along the length asking for the length of the sides of the squares where the cuts were made. Then she would ask for the length of the box. She repeated this process for the width. The students were able to give the correct values but the teacher did not explicitly ask how they got them.

From what they had just done where for each value of the height, they got a corresponding value each for the length and the width, the teacher wanted to find out if the students could recognize that the height was a changing quantity and if they could represent it mathematically. She commented that they could make more boxes aside from those with integral dimensions that most of them made and acknowledged the heights of 2 ½ cm and 4 ½ cm that two groups considered. Then she asked for other possible values of height and some students gave 1 ½ cm and 3 ½ cm. The transcriptions below show how the teacher enabled the students to represent the height.

Speaker	Transcriptions
Teacher	Therefore we can make as many boxes as we want. Which are the only ones that cannot be? Up to what?
Students	Five.
Teacher	If five, what will happen?
Students	There is no more width.
Teacher	So there is no more, box. Since there are many heights, we can represent. So how shall we represent the height since we cannot list as many as we can? What is the representation for height since it can be many?
Students	x
Teacher	1.8, 1.3. What?

Students	x
Teacher	I already heard it. x. If we represent the height by x, how shall we represent the length?

When the teacher asked them to observe the values on the table, the students realized that a relationship existed between the height and the length and between the height and the width. They noted that as the height increased, the length and the width decreased. However, they could not represent these quantities in terms of the height, x even when early on they were able to give their values given a specific value for the height of a box. To help them, she asked them how they obtained 14 cm for length, they said 16 minus 2 and explained that they subtracted from the whole 16, which was the length of the sheet of grid paper. Then they gave 16 minus the height and 16 minus x squared. Eventually, they got 16 minus two times the height. Apparently, early on they just looked at the table on the board or their boxes and so they were able to give the correct values of the length and the width. Apparently also, they did not connect the process of determining these values to the values themselves. Perhaps, they could not represent because they did not reflect on what they did and relate it to the data that they got (Kulm 1994).

In the following transcriptions, the teacher tried to verify if the students could already connect the representation with the process; that is, to give meaning to the representations.

Speaker	Transcriptions
Teacher	So now, what is our representation of the length?
Students	$16 - 2x$.
Teacher	Why is it $16 - 2x$? If x is the height, then why is there 2 there?
Students	Because two sides; Because opposite sides [referring to cutting from the opposite ends of the side of the grid paper associated to the length]
Teacher	Because two. Both sides. Times the length of whatever is the square that you cut, which is the height. How do you represent the width?
Students	$10 - 2x$
Teacher	Why is there a two there again?
Students	Because opposite sides.

The teacher asked what else the students knew about the boxes. They said, volume and area. Considering the volume first, she asked how they should represent it given the representations of the length, width, and height. They were able to give Volume = length x width x height = $(16 - 2x)(10 - 2x)(x) = 4x^3 - 52x^2 + 160$. Then she asked what they called the expression that was on the right side of the equation. They answered trinomial and polynomial. She accepted both and said that it was a polynomial. She asked if there was a relation between x and V. They said yes.

But when she asked what it was, they could not answer. Then she asked “If we assigned one value for x which represents the height, there corresponds how many values for V, the volume?” They responded “one.” Eventually, they were able to identify that the relation was a function. She then asked what they could call the equation. And they said polynomial function. She said that they could then express V as V(x) and said that it is an example of a polynomial function.

Thus, the students were able to represent relationships of changing quantities and give meaning to those representations. Moreover, they were able to use their previous knowledge on polynomials and function to internalize a new concept, polynomial function.

Concluding Remarks

Through lesson study, a group of teachers were able to develop lessons that assessed students’ mathematical thinking as they learned mathematical content. This was because assessment was integrated to teaching. Although they did not have a clear understanding of some concepts and did not verify their conjecture using their empirical data, implementation of the lessons showed that given the opportunity, the students could think on their own in solving an open-ended problem. They could decide on what data they would gather based on their solutions to the problem and represent the relationships shown by these data, mathematically. Lastly, they were able to connect their previous knowledge to learn a concept that was new to them meaningfully.

References

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