PLANNING A LESSON FOR STUDENTS TO DEVELOP MATHEMATICAL THINKING THROUGH PROBLEM SOLVING

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Teaching through problem solving has been emphasized in order to improve the teaching and learning of mathematics. However, it may not be easy for teachers to incorporate problem solving in their classrooms. An ideal way to incorporate problem solving is to plan a lesson and examine it through lesson study. This paper is intended to guide teachers in planning a lesson in which students will develop mathematical thinking through problem solving.

Developing mathematical thinking through problem solving

Teaching mathematics is for students to develop knowledge and skills that are mathematically important both for further study in mathematics and for use in applications in and outside of school is important for school mathematics. However, the objective of mathematics education is not only to enable students to acquire mathematical knowledge and skills but also to foster mathematical thinking. Mathematical thinking is crucial when students acquire and use mathematical knowledge and skills. In other words, students may have a difficult time acquiring and using knowledge and skills unless they have a sufficient ability to think mathematically.

In order to developing mathematical thinking, it is not enough for students simply to receive knowledge and skills by listening to teachers. Students need to actively engage in acquiring knowledge and skills, and to develop mathematical thinking through the process of mathematical activities. Thus students will be able to use these knowledge and skills effectively in their daily life as well as in their future carriers (Brown, 1994).

Based on the above assumption, it is suggested that teachers should provide students with opportunities to acquire knowledge and skills of mathematics through mathematical activities such as problem solving, reasoning and proof, communication, connection, and representation (National Council of Teachers of Mathematics, 2000). To implement such activity -based learning in mathematics classrooms, it is important for teachers to plan lessons that support students acquisition of the knowledge and skills by using mathematical thinking. Many teachers agree that teaching must emphasize the *process* of acquiring mathematics, But teachers often focus solely on teaching the contents to the students rather than providing students with opportunities to actually acquire the contents by using mathematical thinking. One of the reasons for teachers' hesitation to provide

activities that cause students to develop mathematical thinking might be that the teachers have rarely experienced such lessons when they learned mathematics themselves. Moreover, planning lessons that focus more on students' learning process requires teachers to have more knowledge about their students, such as their thinking processes, in addition to having knowledge of the contents of mathematics (Simon & Tzur, 1999).

One of the ways to provide students with an opportunity to acquire not only knowledge and skills but also mathematical thinking is teaching mathematics through problem solving. Teaching mathematics through problem solving has been emphasized for decades, and many reform curriculum materials include problem solving as an integral part of learning mathematics (National Council of Teachers of Mathematics, 1980, 1989, 2000, 2006).

Problem solving in mathematics education is defined as "engaging in a task for which the solution method is not known in advance (National Council of Teachers of Mathematics, 2000)." This means that a problem suitable for problem solving is not necessarily a story problem or a problem in the real world. As long as a student does not know how to solve the problem, it can be a problem for problem solving for the student. In order word, even if a problem is presented as a real world story problem, it might not be a real problem for a student who already knows how to solve the problem. It is now called an exercise.

It is also important to note that teaching through problem solving is more than simply giving a task for students to solve a problem for which they have not learned the solution methods. Table 1. sho ws major differences between teaching of problem solving, which is a simplistic interpretation of the problem-solving approach, which can often be seen in traditional textbooks, and the teaching through problem solving, which is recommended by reform documents such as the NCTM Standards (2000).

Table 1. Problem-Solving Approach

Teaching of problem solving	Teaching through problem solving
• 'What is it? Problem Solving as an approach to develop problem-solving skills and strategies. • How to incorporate it into a curriculum Usually the lessons based on this approach can be found at the end of chapters for developing problem-solving skills and strategies The lesson often end when each student comes up with a solution to the problem. (show and tell)	• 'What is it? Problem solving as a powerful approach for developing mathematical concepts and skills. • How to incorporate it into a curriculum The lessons based on the approach can be found throughout the curriculum in order to develop mathematical concepts, skills, and procedures. The discussion for comparing students' different solutions is important for students to acquire new knowledge and understanding of mathematics.

Using lesson study to incorporate the idea of teaching through problem solving

Planning a lesson for lesson study is always challenging for teachers especially when teachers want to incorporate a new pedagogical idea that they have not experienced before. Lesson study is an ideal way to overcome such a challenging situation, because teachers can work together toward the same goal, which is to understand the new approach and to seek ways to incorporate it into each teacher's classroom (Takahashi & Yoshida, 2004).

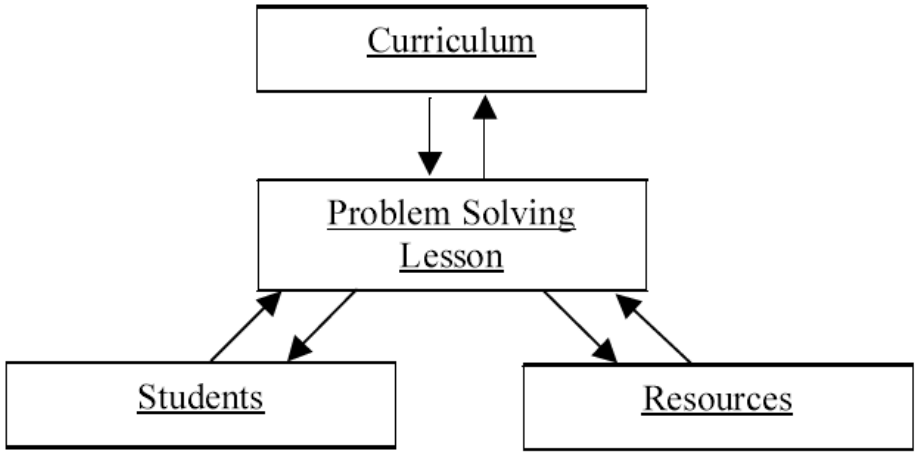
Although there are many different ways to plan a lesson for lesson study, it is often useful to examine each lesson carefully through the following three lenses; curriculum, students, and resources.

- Curriculum
  Any lessons that teachers prepare for their students must be purposeful and meaningful. Although there are many good problems for problem solving, it is important for teachers to identify what mathematics your students are expected to acquire through the lesson. This will become the goal of the lesson. In order to make the goal clear, teachers need to investigate the curriculum on which the students' mathematics learning is based – what they have already learned, what they are expected to learn now, and how their learning now will lead to their future learning. If the lesson is not fit into the curriculum well, it will not be helpful for students to accomplish the goal of the lesson.
- Students
  Knowing students is crucial when teachers plan a lesson. Especially for the lessons that are designed for students to acquire mathematical thinking, teachers need to know how the students might attempt to solve the problem. Without anticipating their students' approaches to the problem, teachers may not be able to plan how to lead the students to develop mathematical thinking by using their informal approaches to solving the problem. It is also important to anticipate students' typical misunderstandings so that teachers can be ready to support the students in overcoming their misunderstandings.
- Resources
  Choosing the best resources is also an important part of planning lessons. These resources include not only good problems in textbooks and other resource materials but also manipulatives, video, and interactive tools on the internet. Moreover, simply knowing what resources are available is not

enough. Teachers should know the potential benefits and drawbacks of each resource. For example, there are problems that are interesting and fun but that may not lead students to develop mathematical thinking at a particular time. When examing the lesson through this lens, teachers might want ask themselves if this is the best resource for students to reach the goal.

In Japanese, the above process of investigating the curriculum, students, and resources is called *Kyozai Kenkyu.* This investigation is important groundwork for planning lessons. The quality of each lesson will greatly rely upon the deepness of *Kyozai Kenkyu. Kyozai Kenkyu* becomes extremely important when teachers plan lessons to address new teaching agenda such as developing mathematical thinking through problem solving.

Figure 1. Planning a Problem Solving Lesson



Planning a lesson for students to develop Mathematical Thinking through Problem Solving

When begin to plan a problem-solving lesson, there are multiple entry points. For example teachers might want to begin *Kyozai Kenkyu* by carefully examining the curriculum to identify what mathematics the students are expected to acquire. Another entry point is to begin with examining students' work to identify what might be the area where students need to deepen their understanding in order to improve their mathematical ability. The third entry point is by examining resources to plan a problem-solving lesson. Although planning lessons in all three ways is encouraged for teachers, it is important to closely look at the lesson planning process by using the third entry point, resources, because many novice teachers take a wrong pass after they chose an attractive problem.

When teachers read teacher resource books and textbooks, and participate in professional development workshops and conferences, they often find an interesting and fun problem for students. Although these problems are useful resources, it is important to note that they are raw material and need to be prepared for a problem-solving lesson. It is not a good idea for teachers to bring those raw materials to classrooms to simply ask students to solve them or to show students how to solve problems. When teachers do not have a clear goal of the lesson, the lesson often become meaningless for the students.

At the APEC Tokyo/Sapporo symposia, Stacey (2006) used an interesting website, "Crystal Ball[1]", to illustrate the processes of mathematical thinking in the context of a problem solving lesson. If teachers are inspired by her talk and want to use the website to plan a problem-solving lesson for their own students to develop mathematical thinking, what should they do?

Investigating the problem

As Stacey (2006) describes, there are several ways to find out the trick. It is important for teachers to attempt several different approaches to discover mathematics behind the "Crystal Ball". In this particular case, teachers want to spend time to find the trick by themselves. Then, they should look at the same web page from students' viewpoints, asking themselves how could they find out the trick if they were a fifth grade student or a seventh grade student? This will lead teachers to investigate the problem through the student lens, although the investigation originally began through resource lens. After trying to figure out find out the trick in various ways, it might be a good idea to compare all the approaches for figuring out the trick to see how these approaches are related and how they are different. What mathematical knowledge and skills, and mathematical thinking are required for each approach?

Through this investigation, a group of teachers might be able to come up the following conclusion.

1 http://www.cyberglass.biz/customflash/ghostwhisperer/

In order to find out the trick, one of the approaches is to try several specific examples to find a pattern among the examples. Students typically use this inductive approach and find out that there might be mechanism behind the trick, but it is difficult to figure out why the pattern exists. Another approach is to investigate the process of calculations described in the “Crystal Ball” instruction in order to find out what calculations are actually carried out to get the symbol that you need to imagine. This deductive approach demands that students write, interpret, and use mathematical expressions to investigate the trick, then find out why the crystal ball gives you the same symbol no matter what two digit numbers are chosen. During this investigation, students will be using their previous learning of the properties of the basic operations, the notion of place value, and the use of symbols in mathematical expressions to see the generalized pattern.

Investigating the problem through other lenses

The next step toward planning a lesson by using the “Crystal Ball” might be to narrow down what mathematics teachers expect students to acquire through this problem solving. From previous investigation, teachers agree that most students should be able to try at least a couple of specific cases to draw a conclusion that there might be a trick behind the website. Moreover, some of the students might be able to find that the procedure that the “Crystal Ball” gives always produces a number that is a multiple of nine. It is, however, expected that many students might not be able to figure out why the procedure always gives a number that is multiple of nine, because it requires students to manipulate mathematical expressions.

The above process describes how teachers can investigate the problem through the lens of "students". The next step might be the investigation through the "curriculum" lens. In the *Curriculum focal points for pre -kindergarten through grade 8 mathematics: a quest for coherence* (National Council of Teachers of Mathematics, 2006), one of the focal points in the middle school is to write, interpret, and use mathematical expressions and equations to solve problems. It is expected that students become able to

1) write mathematical expressions and equations that correspond to given situations,
2) evaluate expressions, and
3) use expressions and formulas to solve problems.

One of the challenges for the students is to write mathematical expressions that correspond to a given situation. Sometimes students may be reluctant to write mathematical expressions because they often try to find the answer by simply carrying out calculations and cannot see the merits of writing mathematical expressions. In order to overcome students’ reluctance to write mathematical expressions, therefore, it is important that they learn how writing mathematical expressions that can help them to solve problems.

Form that discussion, the goal of this problem-solving lesson might be to provide students an opportunity to learn inductive reasoning by writing, interpreting, and using mathematical expressions.

Designing the flow of the lesson

After going through the ground work, *Kyozai Kenkyu*, the group of teachers should move toward actually discussing how to pose the problem, and what questions a teacher can ask the students toward acquiring mathematical knowledge and skills. There are several types of lesson plans for lesson study. One important component that most Japanese lesson plans share is the section called "anticipated students' responses." The quality of the section of student anticipated responses relies heavily on the richness of *Kyozai Kenkyu*. Moreover, this section usually contributes greatly to the quality of the discussion that a teacher will be leading after students present their various solutions.

It is often a good idea for teachers to prepare answers for the following questions to develop a short sketch of the lesson.

- Purpose of the problem solving (goal of the lesson)
  What mathematics, beside developing problem solving skills, would you teach by using this situation?
- Questioning
  How would you pose the problem?
  What question(s) would you ask of your students for them to learn mathematics?
- Beyond show and tell
  Anticipate students' responses to your questions, including misunderstandings, to facilitate discussion.
  Briefly describe how you would facilitate discussion.

The Appendix shows an example of the lesson plan for the problem-solving lesson using the "Crystal Ball."

Conclusion

Planning lessons for lesson study demands that teachers spend time and effort. Although it is time consuming, once teachers experience this process with their colleagues, they start seeing their everyday lessons differently. It is not easy for teachers to change their teaching practice in a short time. However, it will be a great step toward addressing mathematical thinking in their everyday lessons.

It will be also a powerful experience for teachers to observe an actual classroom based on the lesson plan that the group planned together. Since the mathematical thinking can be observed mostly in the process of students' problem solving and dialogues among students, the entire process of lesson study is expected to improve mathematical thinking.

1 http://www.cyberglass.biz/customflash/ghostwhisperer/

References

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8TH GRADE MATHEMATICS LESSON PLAN

April 26, 2007
Las Cruces, NM
Instructor: Akihiko Takahashi

1. Title of the lesson: The Secret of The Crystal Ball
2. Goal of the lesson:
    1. To deepen students' understanding of the properties of the basic operations and place value by writing, interpreting, and using mathematical expressions through problem solving
    2. To help students become good problem solvers by
        i. encouraging them to use their prior knowledge to examine a problem situation in order to develop their ability to use logical reasoning to make conjectures, and
        ii. encouraging them to examine and justify the conjectures presented by their peers in order to find a solution to the problem.
    3. Provide opportunities for students to recognize the importance of working with their peers in order to deepen their understanding of mathematics

3. Instruction of the Lessons

In the Curriculum focal points for pre-kindergarten through grade 8 mathematics: a quest for coherence (National Council of Teachers of Mathematics Inc. Reston VA., 2006), one of the
focal points in the middle school is to write, interpret, and use mathematical expressions and equations to solve problems. It is expected that students become able to:

1) write mathematical expressions and equations that correspond to given situations,
2) evaluate expressions, and
3) use expressions and formulas to solve problems.

One of the challenges for the students is to write mathematical expressions that correspond to a given situation. Sometimes students may be reluctant to write mathematical expressions because they often try to find the answer by simply carrying out calculations and cannot see the merits of writing mathematical expressions. In order to overcome students' reluctance to write mathematical expressions, therefore, it is important that they learn how writing mathematical expressions can help them to solve problems.

When designing such problem-solving lesson, it is important to keep in mind that solving a problem is a process for providing an opportunity for students to appreciate that writing, interpreting, and using mathematical expressions. Therefore, the flow of the lesson should not solely focus on finding the correct answer, but also the process of solving the problem.

This lesson is designed for students' to understand how writing, interpreting, and using mathematical expressions help them analyze the problem situation and empower them to solve a problem.

The problem for this lesson to figure out the mechanism behind a trick named "Crystal Ball" from the website of a popular TV program, Ghost Whisperer (http://www.cbs.com/primetime/ghost_whisperer/crystal ball.shtml). The website is based on a popular math trick and use Flash, multimedia authoring program for web applications, to make it interactive and engaging. The procedures that described on the website is

> *Chose any two digit number, add together both digits and then subtract the total from your original number. When you have the final number look it up on the chart and find the relevant symbol. Concentrate on the symbol and when you have it clearly in your mind click on the Ghost Whisperer crystal ball and it will show you the symbol you are thinking of*

In order to find out the trick, one of the approaches is to try several specific examples to find a pattern among the examples. Students typically use this inductive approach and find out that there might be mechanism behind the trick but it is difficult to figure out why the pattern exists. Another approach is to investigate the process of calculations described in the "Crystal Ball" instruction in order to find out what calculations are actually carried out to get the symbol that you need to imagine. This deductive approach demands that students write, interpret, and use mathematical expressions to investigate the trick, then find out why the crystal ball always gives you the same symbol no matter what two digit numbers are chosen. During this investigation, students will be using their previous learning of the properties of the basic operations, the notion of place value, and the use of symbols in mathematical expressions to see the generalized pattern.

4) Flow of the Lesson

Learning Activities, Teacher's Questions and Expected Students' Reactions	Teacher's Support	Points of Evaluation
1. Introduction to the Problem By experiencing the "Crystal Ball" on the internet, students will become familiar with the site. 1. Chose any two digit number, 2. Add together both digits, 3. Subtract the total from your original number 4. When you have the final number look it up on the chart and find the relevant symbol. 5. Concentrate on the symbol and when you have it clearly in your mind 6. Click on the crystal ball to see the symbol	Ask a couple of volunteer students to try the website so that all the students understand the procedures described on the webpage. Help students to see the website always gives you the relevant symbol.	Do students understand the procedure? Do students see what is happening on the website?
2. Posing the problem By asking the following question, engage students to find the trick behind the "Crystal Ball" webpage. With which opinion do you agree? a. It is just a coincident and there is nothing special in the "Crystal Ball" webpage. b. There might be a trick behind the "Crystal Ball". c. The "Crystal Ball" webpage actually reads your mind. Let's find the trick behind the "Crystal Ball" webpage!	Each student will be working with his/her partner to find a trick by using their prior knowledge. Provide students with worksheets to keep their work for the whole class discussion.	Do students see there must be a trick behind the "Crystal Ball" webpage

3. Problem Solving Working with a partner, students try to find the trick behind the "Crystal Ball" webpage. Anticipated students' responses: a. Try a couple of specific examples to notice that the relevant symbol might be always the same but do not know why these symbols are the same. b. By examining several specific examples, he/she realizes that the final number will always be a multiple of nine, and the symbols on the chart that correspond to multiple of nine are all the same. However, he/she does not know why the final number will always be a multiple of nine. c. Write, interpret, and use mathematical expressions to investigate the trick □ $a\ b$ as a chosen two digit number □ The value of $a\ b$ is $10a + b$ □ Write a mathematical expression to express the procedure $(10a + b)(a + b)$ $= 10a + ba\ b$ $= 10aa + bb$ $= 9a$ Therefore the final number will always be a multiple of nine	Encourage students to try at least a couple of specific examples. Help students understand that methods (a) and (b) may not be able to answer the question why all the final numbers give you the same symbol. Encourage students to investigate the process of calculations described in the instructions to "Crystal Ball" in order to find out what calculations are actually carried out to get the symbol that you need to imagine.	Do students try at least a couple of specific examples to notice that the relevant symbol from your calculation might always be the same.
4. Discussing Students' Solutions (1) Ask students to explain their solutions to the other students in the class. (2) Facilitate students' discussion about their solutions, then lead them to understand that writing, interpreting, and using mathematical expressions helped them understand the trick behind the "Crystal Ball" webpage.	Write students' solutions and ideas on the blackboard in order to help students understand the discussion.	Can students explain their solutions to their peers? Can students examine and justify the solutions presented by their peers?
5. Summing up (1) Using the writing on the blackboard, review what students learned through the lesson. (2) Ask students to write a journal entry about what they learned through this lesson.	Encourage students to use the writing on the board as a reference when they write the journal entry.	

Reference

National Council of Teachers of Mathematics Inc. Reston VA. (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence.*

http://www.cbs.com/primetime/ghost whisperer/crystal ball .shtml

This Lesson Plan is prepared for the Lesson Study Workshop at Las Cruces, NM. April 26, 2007
By Akihiko Takahashi

Board writing Plan for the "Crystal Ball"

The Crystal Ball

1. Chose any two digit number,
2. Add together both digits,
3. Subtract the total from your original number
4. When you have the final number look it up on the chart and find the relevant symbol
5. Concentrate on the symbol and when you have it clearly in your mind click on the crystal ball to see the symbol

What is happening on the website?
Use the worksheet to figure out

Case 1: 56
5+6=11
56-11=45 Symbol A

Case 2: 78
7+8=15
78-15=63 Symbol A

Students' approach A

Students' approach B

Students' approach C

The Crystal Ball always give you the same symbol no matter what two digit numbers are chosen because

1. the final number that the procedure give by the Crystal Ball always be a multiple of 9,
2. the symbols on the chart that correspond to multiple of 9 are all the same. (with two exceptions, 90 and 99)

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