SETTING LESSON STUDY WITHIN A LONG-TERM FRAMEWORK OF LEARNING

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Lesson Study is a format to build and analyse classroom teaching where teachers and researchers combine to design lessons, predict how the lessons might be expected to develop, then carry out the lessons with a group of observers bringing multiple perspectives on what actually happened during the lesson. This article considers how a lesson, or group of lessons, observed as part of a lesson study may be placed in a long-term framework of learning, focusing on the essential objective of improving the long-term learning of every individual in classroom teaching.

INTRODUCTION

This paper began as a result of a participation in a lesson study conference (Tokyo & Sapporo, December 2006) in which four lessons were studied as part of an APEC (Asian and Pacific Economic Community) study to share ideas in teaching and learning mathematics to improve the learning of mathematics throughout the communities. It included the observation of four classes (here given in order of grade, rather than order of presentation):

Placing Plates (Grade 2)
December 2nd 2006, University of Tsukuba Elementary School
- Takao Seiyama

Multiplication Algorithm (Grade 3)
December 5th 2006, Sapporo City Maruyama Elementary School
- Hideyuki Muramoto

Area of a Circle (Grade 5)
December 2nd 2006, Universty of Tsukuba Elementary School
- Yasuhiro Hosomizu

Thinking Systematically (Grade 6)
December 6th 2006, Sapporo City Hokuto Elementary School
- Atsutomo Morii

The objective of this paper is to set these classes within a long -term framework of development outlined in Tokyo at the conference (Tall, 2006), which sets the growth of individual children within a broader framework of mathematical development. Long-term the development of individual children depends not only on the experiences of the lesson, but in the experiences of the children prior to the lesson and how experiences 'met-before' have been integrated into their current knowledge framework.

In general, it is clear that lesson study makes a genuine attempt:

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to design a sequence of lessons according to well-considered objectives;
to predict what may happen in a lesson;

to have a group of observers bring multiple perspectives to what happened, without prejudice; and ultimately

to improve the teaching of mathematics for all.

Lesson study is based on a wide range of communal sharing of objectives. At the meeting I was impressed by one essential fact voiced by Patsy Wang -Iverson:

> The top eight countries in the most recent TIMMS studies shared a single charactistic, that they had a smaller number of topics studied each year.

Success comes from focusing on the most generative ideas, not from covering detail again and again. This suggests to me that we need to seek the generative ideas that are at the root of more powerful learning.

For many individuals, mathematics is *complicated* and it gets more complicated as new ideas are encountered. For a few others, who seem to grasp the essence of the ideas, the *complexity* of mathematics is fitted together in a way that makes it essentially *simple* way. My head of department at Warwick University in the sixties, Sir Christopher Zeeman noted perceptively:

> "Technical skill is a mastery of complexity, while creativity is a mastery of simplicity" (Zeeman, 1977)

This leads to the fundamental question:

> How can we help *each and every child* find this simplicity, in a way that works, *for them?*

Lesson study focuses on the *whole class activity*. Yet within any class each child brings differing levels of knowledge into that class, related not only to what they have experienced before, but how they have made connections between the ideas and how they have found their own level of simplicity in being able to think about what they know.

To see simplicity in the complication of detail requires the making of connections between ideas and focusing on essentials in such a way that these simple essentials become generating principles for the whole structure.

In my APEC presentation in Tokyo (Tall 2006), I sought this simplicity in the way that we humans naturally develop mathematical ideas supported by the shared experiences of previous generations. I presented a framework with three distinct worlds of mathematical development, two of which dominate development in school and the third evolves to be the formal framework of mathematical research. The two encountered in school are based on (conceptual) embodiment and (proceptual) symbolism. I described these technical terms in more detail in Tall (2006) and in a range of other recent papers on my website (www.davidtall.com/papers).

Essentially, conceptual embodiment is based on human perception and reflection. It is a way of interacting with the physical world and perceiving the properties of objects and, through thought experiments, to see the essence of these properties and begin to verbalise them and organize them into coherent structures such as Euclidean geometry. Proceptual symbolism arises first from our *actions* on objects (such as counting, combining, taking away etc) that are symbolized as concepts (such as number) and developed into symbolic structures of calculation and symbolic manipulation through various stages of arithmetic, algebra, symbolic calculus, and so on. Here symbols such as 4+3, $x^2 + 2x + 1$, $\int \sin x \, dx$ all dually represent processes

to be carried out (addition, evaluation, integration, etc) and the related concepts that are constructed (sum, expression, integral, etc). Such symbols also may be represented in different ways, for instance 4+3 is the same as 3+4 or even ‘1 less than 4+4’ which is ‘1 less than 8’ which is 7. This flexible use of symbols to represent different *pro*cesses for giving the same underlying con*cept* is called a *procept.*

These two worlds of (conceptual) embodiment and (proceptual) symbolism develop in parallel throughout school mathematics and provide a long-term framework for the development of mathematical ideas throughout school and on to university, where the

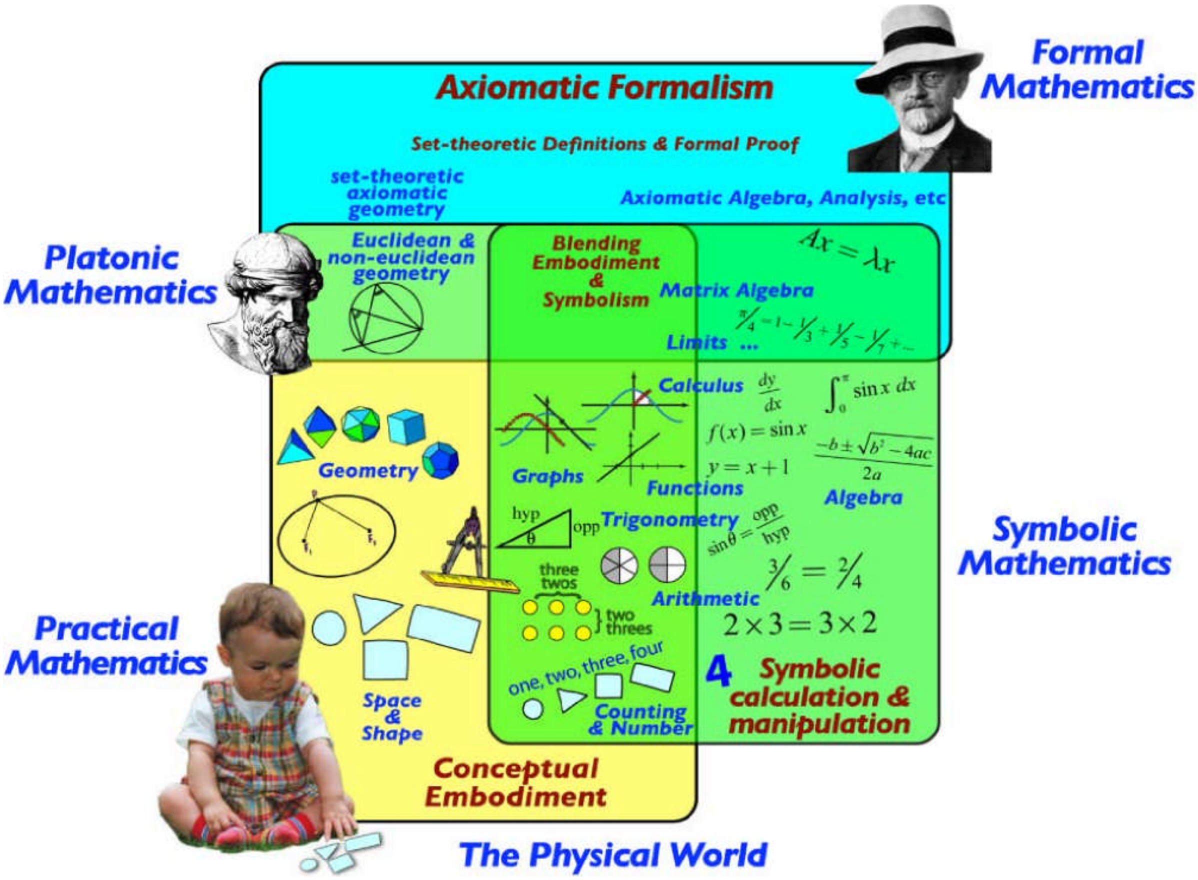

Figure 1. The three mental worlds of (conceptual) embodiment, (proceptual) symbolism and (axiomatic) formalism

focus changes to the formal world of set-theoretic definition and formal proof.

In figure 1 we see an outline of the huge *complication* of school mathematics. On the left is the development of conceptual embodiment from practical mathematics of physical shapes to the platonic methods of Euclidean geometry. In parallel, there is a development of symbolic mathematics through arithmetic, algebra, and so on, with the two blending as embodiment is symbolized or symbolism is embodied.

The long-term development begins with the child's perceptions and actions on the physical world. In figure 1 the child is playing with a collection of objects: a circle, a triangle, a square, and a rectangle. The child has two distinct options, one to focus on his or her *perception* of each object, seeing and feeling their separate properties, the other is through *action* on the objects, say by counting them: one, two, three, four.

The focus on perception, with vision assisted by touch and other senses to play with the objects to discover their properties, leads to a growing sense of space and shape, developing through the use of physical tools—ruler, compass, drawing pins, thread—to enable the child to explore geometric ideas in two and three dimensions, and on to the mental construction of a perfect platonic world of Euclidean geometry. The focus on the essential qualities of points having location but no size, straight lines having no width but arbitrary extensions and on to figures made up using these qualities leads the human mind to construct mental entities with these essential properties. Platonism is a natural long-term construction of the enquiring human mind.

Meanwhile, the focus on action, through counting, leads eventually to the concept of number and the properties of arithmetic that benefit from blending embodiment and symbolism, for example, 'seeing' that 2 x 3= 3 ! 2 by visualizing 2 rows of 3 objects being the same as 3 columns of 2 objects. Long-term there is a development of successive number systems, fractions, rationals, decimals, infinite decimals, real numbers, complex numbers. (What seems to the experienced mathematician as a steady extension of number systems is, for the growing child, a succession of changes of meaning which need to be addressed in teaching. We return to this later.)

The symbolic world develops through whole number arithmetic, fractions, decimals, algebra, functions, symbolic calculus, and so on, which are given an embodied meaning through the number-line, Cartesian coordinates, graphs, visual calculus, with aspects of the embodied world such as trigonometry being realized in symbolic form. In the latter stages of secondary schooling, the learner will meet more sophisticated concepts, such as symbolic matrix algebra and the introduction of the limit concept, again represented in both embodied and symbolic form.

The fundamental change to the formal mathematics of Hilbert leads to an axiomatic formalism based on set-theoretic definitions and formal proof, including axiomatic geometry, axiomatic algebra, analysis, topology, etc.

Cognitive development works in different ways in embodiment, symbolism and formalism (Figure 2). In the embodied world, the child is relating and operating with perceived objects (both specific and generic), verbalizing properties and shifting from practical mathematics to the platonic mathematics of axioms, definitions and proofs.

In the symbolic world, development begins with actions that are symbolized and coordinated for calculation and manipulation in successively more sophisticated contexts. The shift to the axiomatic formal world is signified by the switch from concepts that arise from perceptions of, and actions on, objects in the physical world to the verbalizing of axiomatic properties to define formal structures whose further properties are deduced through mathematical proof.

Focusing on the framework appropriate to school mathematics, we find the main structure consists of two parallel tracks, in embodiment and symbolism, each building on previous experience (met-befores), with

> *embodiment* developing through perception, description, construction, definition, deduction and Euclidean proof after the broad style suggested by van Hiele;
>
> *symbolism* developing through increasingly sophisticated compression of procedures into procepts as thinkable contexts operating in successively broader contexts.

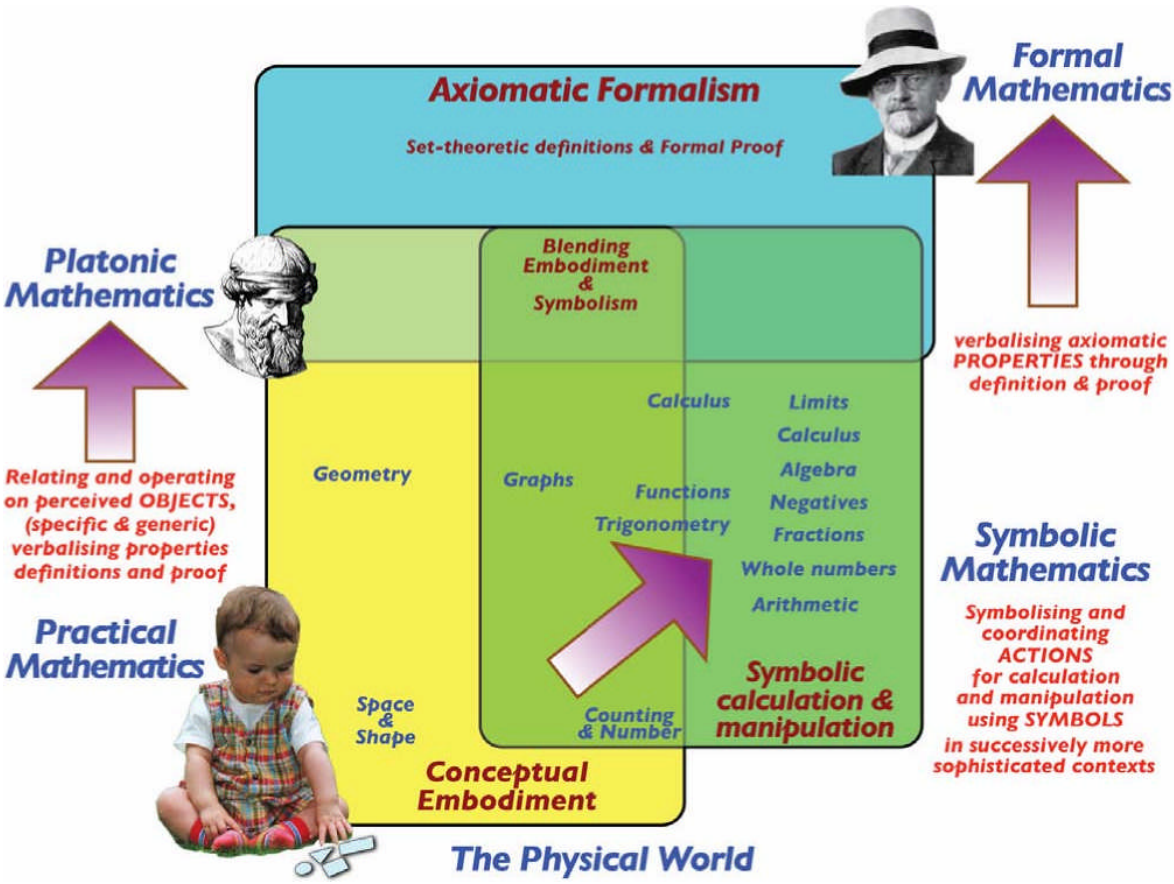

Figure 2: long-term developments in the three worlds

These two developments are fundamentally different. On the one hand, embodiment gives a global overall picture of a situation while symbolism begins with coordinating actions, practicing sequences of actions one after another to build up a procedure, perhaps refining this to give different procedures that are more efficient or more effective, using symbolism to record the actions as thinkable concepts. The problem here is that the many different procedures can, for some, seem highly complicated and so the teacher faces the problem of reducing the complexity, perhaps by concentrating on a single procedure to show the pupils what to do, without becoming too involved in the apparent complications. Procedures, however, occur *in time* and become routinized so that the learner can *perform* them, but is less able to *think about* them. (Figure 3.)

As an example, consider the teaching of long-multiplication. First children need to learn their tables for single digit multiplication from 0 ! 0 to 9 ! 9. They also need to have insight into place value and decimal notation.

The method used by Hideyuki Muramoto in the lesson study at Sapporo City Maruyama Elementary School on December 6, 2006 can be analysed in terms of an initial embodiment representing 3 rows of 23. Here the learner can *see* the full set of counters: the problem is how to *calculate* the total. The embodiment can be broken down in various ways, separating each row into subsets appropriate to be able to compute the total. In the previous lesson the students had already considered 3 rows of 20 and had broken this into various sub-combinations, breaking each row into 10+10 or 5+5+5+5, or even 9+9+2, or 9+2+9. Now the problem related to breaking

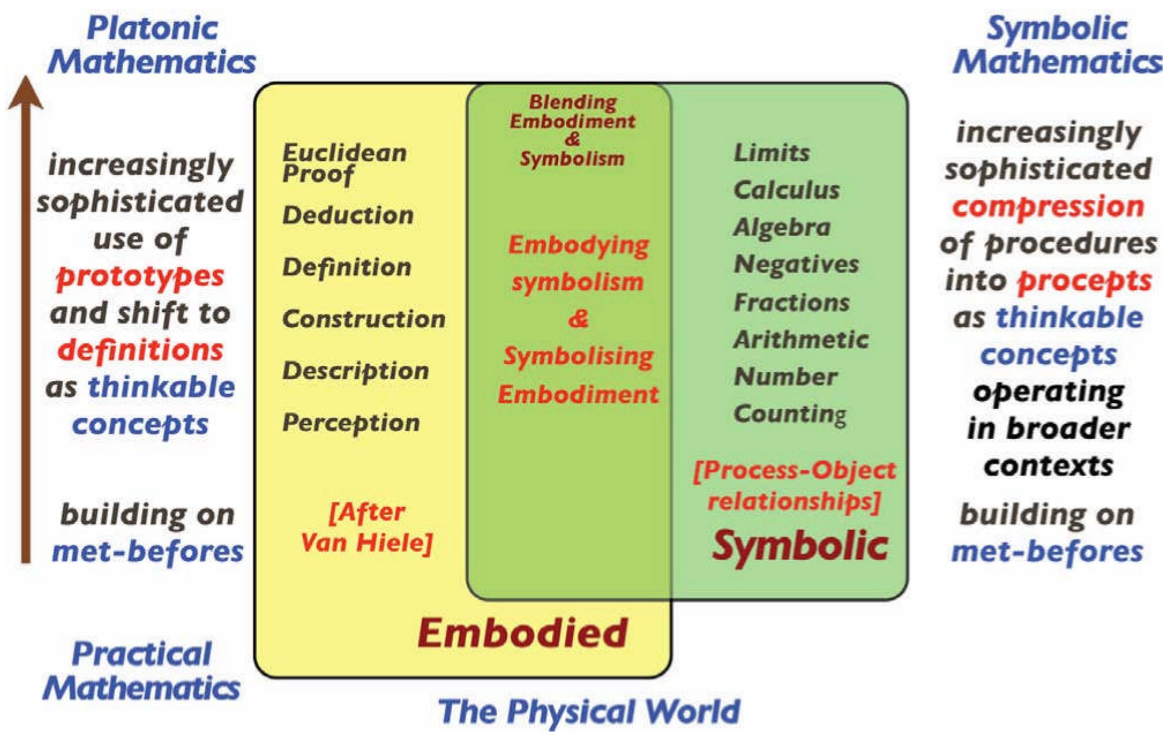

Figure 3: Developmental framework through embodiment and symbolism

23 into sub-combinations, suggested possibilities included 10+10+3 and 9+5+9 (but not 5+5+5+5). Three lots of 10+10+3 gives 30+30+9, which easily gives 60+9, which is 69. Three lots of 9+5+9 is more difficult requiring the sum 27+15+27. Here we have two different procedures giving the same result, 69, and the most productive way forward is to break the number 23 into tens and units and multiplying each separately by 3.

In this analysis, the embodiment gives the *meaning* of the calculation of a single digit times a double digit number, while the various distinct sub-combinations give different ways of *calculation*, from which the sub-combination as tens and units is clearly the simplest and the most efficient.

The approach has a general format:

1. *Embody the problem* (here the product 23 ! 3);
2. *Find several different ways of calculation* (here 23 ! 3 is three lots of 10+10+3 or three lots of 9+9+5) *where the embodiment gives meaning to symbolism;*
3. See *flexibility*, that all of these are the same;
4. See the *standard algorithm is the most efficient.*

Thus embodiment gives meaning while symbolism enables compression to an efficient symbolic algorithm.

It may be that not all the children in the class will be able to cope with the different procedures (for instance, one would expect the suggestion 9+5+9 to come from a more able child and the computation would not be easy for some). Thus, the dynamic of the whole class may not be shared by all individuals. The more successful may see the different ways of computing the result as different procedures with the same

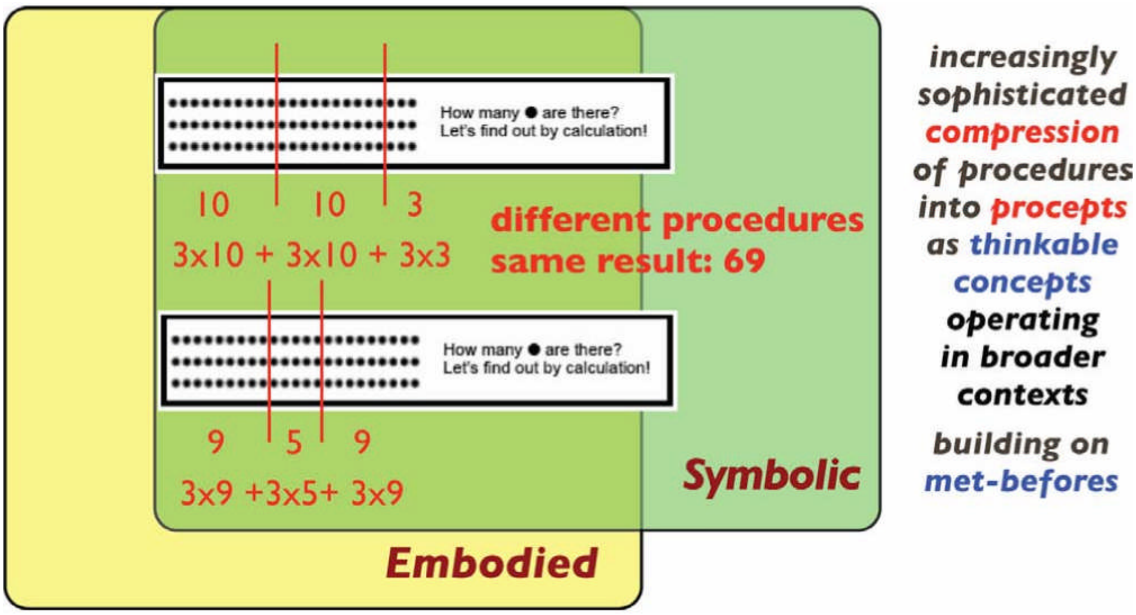

Figure 4: multi-digit arithmetic from embodiment to symbolism

effect, and meaningfully see that the standard algorithm is just one of many that is chosen because it is efficient and simple. They may sense that it is not appropriate to use a more complicated method like 3 times 9+9+5 and not even desire to carry it through without this compromising their insight that different procedure s can give the same result. Meanwhile, those who are less fluent in their tables may feel insecure and seek an easy method to cope that is less complicated. A single procedure may have its attractions, showing *how to do it,* without the complication of *why it works.* It may have attractions to the teacher to teach the method by rote as this may have short-term success without extra complication.

In this way, the same lesson may be seen very differently by different participants, at one extreme, a great insight into the meaning and construction of the standard algorithm within a rich conceptual framework, at another extreme, a great deal of complication and a desire to cope by seeking a procedure *that works* rather than a situation which is too complicated to understand. This bifurcation is what Gray & Tall (1994) called the *proceptual divide* between those who seek to maintain procedures that work at the time rather than flexible methods that require many meaningful connections in a broader knowledge structure.

BLENDING KNOWLEDGE STRUCTURES IN THE BRAIN

In addition to this combination of embodiment and symbolism to give meaning to number concepts and operations, there are subtle features of successive number systems that cause additional problems. A mathematician may see successive numbers systems such as:

Whole Numbers

Fractions

Rational Numbers

Positive and Negative numbers

Real Numbers consisting of rationals and irrationals

as a growing extension of the number system. They can all be marked on an (embodied) number line and the child should be able to *see* how each one is extended to the next. However, for the learner, each extension has subtle aspects which can cause significant problems. We all know of the difficulty of introducing the concept of fraction and of the problem of multiplying negative numbers. There are subtle difficulties between counting and measuring:

Counting 1, 2, 3, ... has successive numbers, each with a next number and no numbers in between. Multiplying these numbers gives a bigger result ... etc.

Measuring numbers are continuous without a ‘next’ number and have fractions between. Multiplying can give a smaller result.

Elsewhere (e.g. Tall, 2007), I use the idea of *conceptual blending* from Fauconnier & Turner (2003) to shed light onto the cognitive strengths and difficulties of long -term learning in mathematics. Fauconnier and Turner share the distinction of being the first cognitive scientists to integrate the fundamental ideas of *compression* and *blending* of knowledge into a single framework. In considering how students learn long-term, this suggests we need to be aware not only what experiences students have had before, but how they compress this experience into thinkable concepts and how different knowledge structures are blended together to produce new knowledge.

USING A LONG-TERM FRAMEWORK OF EMBODIMENT AND SYMBOLISM IN LESSON STUDY

Putting together the ideas of growth in elementary mathematics discussed here and in the earlier paper (Tall, 2006), we find that the parallel development of embodiment and symbolism suggests:

> Embodiment gives human meaning as prototypes, developing verbal description, definition, deduction.
>
> Symbolism is based initially on human action, leading to symbol use, either through procedural learning or through conceptual compression to flexible procept.
>
> Experiences build met-befores in the individual mind that are used later to interpret new situations.
>
> Different experiences may be blended together, requiring a study of what learners bring to a new learning experience.

Tall (2006) also observed:

> Embodiments may work well in one context but become increasingly complex; flexible symbolism may extend more easily.

This means that successful students may show a long-term tendency to shift to symbolism to work in a way that is both more powerful and (for them) more simple.

In our earlier discussions in Tokyo, great emphasis was made not only on meaningful learning of mathematical concepts and techniques, but also on *problem-solving* in new contexts. Learning new concepts can be approached in a problem-solving manner. My own view is that learners must take responsibility for their own learning, once they have the maturity to do so, which includes developing their own methods for solving problems. I also believe that teachers have a duty, as mentors, to help focus students on methods that are powerful and have long-term value.

In studying lessons, therefore, we need some objectives to consider. There are so many theories in the literature, from Bruner’s (1966) analysis into enactive iconic and symbolic, Fischbein’s (1987) categorization into intuitive, algorithmic and formal, the Pirie-Kieren theory (1994) with its ideas of ‘making’ and ‘having’ images and successive levels of operation, Dreyfus and colleagues RBC theory (Recognising, Building-With, Consolidating), theories of problem-solving (Schoenfeld 1985, Mason *et al.* 1982) and so on. With such a wealth of ideas to choose from and build on (and build with), I will hear focus on three simple ideas that are important. You may choose different ones, but in the long run, it is important for those studying lessons to have principles with which they are working and a fundamental framework for each lesson study. I suggest the need in long-term development to focus on three aspects:

Building thinkable concepts in (*meaningful)* knowledge structures;

Using knowledge structures in *routine* and *problem* situations (where 'routine' includes practising for fluency);

Proving knowledge structures (*as required in context).*

I would see these three aspects being applied *before, during* and *after* each lesson.

BEFORE: What is the purpose of the lesson

(e.g. **Building** new constructs, **Using** *known routines* or *problem-solving*, **Proving** in some sense) and what concepts may the learners have in mind that may be used in the lesson? (*met-befores, blends, routines, problem-solving techniques)*

DURING: How do learners use their knowledge structures during the lesson to make sense of it? (*met-befores, blends, routines, problem-solving techniques)*

AFTER: What knowledge structures are developing that may be of value in the future? (*met-befores, blends, routines, problem-solving techniques)*

LESSONS STUDIES

Four classes were videoed during our previous meeting in Japan, December, 2006.

Placing Plates (Grade 2)
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My purpose is to focus on the role of these lessons in long-term learning, and to consider how the long-term development of each and every student may be affected by the lesson within the framework suggested above.

There is already a great deal of evidence of the use of broad principles in the planning of the lessons which are formulated in the lesson plans. Taking a few quotes at random we find:

The goal of the Mathematics Group at Maruyama is to develop students ability to use what they learned before to solve problems in the new learning situations by making connections.

In addition, we want to provide 3rd grade students with experiences in mathematics that enable them to use why they learned before to give problems in new learning situations by making connections.

Through teaching mathematics, I would like my students to develop 'secure ability' for finding problems on their own, studying by themselves, thinking, making decisions, and executing those decisions. Moreover, I would like to help my students like mathematics as well as enjoy thinking.

In order for students to find better ideas to solve the problem, it is important for the students to have an opportunity to feel that they really want to do so.

Starting in April (beginning of the school year), I taught the students to look at something from a particular point of view such as 'faster, easier, and accurate' when they think about something or when they compare something.

If you think about the method that uses the table form this point of view, students might notice that "it is accurate but it takes a long time to figure out: or "it is accurate but it is complicated."

In order to solve a problem in a short time and with less complexity, it is important for the students to notice that calculation using a math sentence in necessary.

Each of these shows a genuine desire for students to make connections, to rely on themselves for making decisions and to seek more powerful ways of thinking with less complexity. The videos of the classes themselves show high interaction between the students, and with the teacher, carefully orchestrated by the teacher to bring out essential ideas in the lesson.

We now briefly look at each lesson in turn, to see how it fits with a long-term development blending embodiment and symbolism, what aspects of Building, Using, and Proving arise as an explicit focus of attention, before, during, and after the lesson. In particular, we need to look deeper at how individual children respond to the lesson in ways that may be appropriate for their long-term development of powerful mathematical thinking.

In the pages which follow, I reproduce overheads from my presentation that look at each of the lessons to see where it fits in the overall plan of building ideas from a blend of embodiment and symbolism to build use and prove powerful mathematical concepts. This is, in no way, intended to be a once-and-for-all analysis. It is offered as a preliminary analysis for those developing lesson study to initiate discussion on how to implement the techniques of lesson study within a long-term framework that focuses on improving the learning of mathematics for each and every student.

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In Britain, attention is turning to the needs of ‘pupils at risk’ who need extra support and to the ‘gifted and talented’ who need extra challenges.

> É for pupils at risk of falling behind, early intervention and special support to help them catch up. This is already underway with the ‘Every Child a Reader’ programme for literacy, which is now being matched with the ‘Every Child Counts’ initiative for numeracy, alongside one-to-one tuition for up to another 600,000 children.Gordon Brown, The Guardian, May 15, 2007

However, it is not a linear race, with some ‘falling behind’ and others ‘racing ahead’. It is also a question of different kinds of learning and different ways of coping.

Assuming our major purpose is to improve the long-term learning of mathematics for each and every one of our children, I suggest that there is a need for lesson study to be placed in a long-term framework to design and monitor the long-term development of individuals, to gain insight not only what needs to be learnt and how, but also why some develop flexible, powerful mathematical thinking and others have serious difficulty.

The framework offered is based on the different styles of cognitive growth in embodiment and symbolism over the long -term, and the way in which different individuals build on mental structures based on ideas met-before.

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