MATHEMATICAL THINKING AND THE ACQUISITION OF FUNDAMENTALS AND BASICS

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1. Fundamentals and Basics in Mathematics Education

The initial report of the Central Education Council issued in 1996 proposed that the formation of the "power for living," whereby one learns and thinks for oneself, in an educational environment free of pressure is the basis of education in the 21st century. Subsequently, based on a number of additional reports, the final report of the Central Education Council was issued, and the curriculum created accordingly along school curriculum guidelines went into effect in 2002. Under this curriculum, educators are required to help pupils form the capacity to identify issues on their own and proactively solve problems amid a society undergoing rapid change. According to school curriculum guidelines, fundamentals and basics are the basis that supports the various living and learning activities of children. For example, they are the basis of daily living activities, various school activities, the continuous learning of mathematics, and future social and lifelong activities. The guidelines emphasize the systematic acquisition of fundamentals and basics, through repeated study if needed, in order to enable the smooth pursuit of such activities.

The above approach to fundamentals and basics suggests a large number of issues related to current education and study guidance in the field of elementary school mathematics, such as the following:

A. An approach that combines guidance that emphasizes fundamentals and basics and nurtures individuality by enabling children to learn and think for themselves
B. Fundamentals and basics should be understood simply as formal guidance in terms of knowledge and skills, but they should include the aim of guidance in terms of the ability to think, judge, express oneself, and so on.
C. Activities to acquire fundamentals and basics should be understood in terms of children achieving goals through the autonomous study of problem solving, and fundamentals and basics should be considered to include study methods and the ability to solve problems.

The formation of the ability to learn and think independently should be understood in terms of the fundamentals and basics that make up the core that develops the drive to learn independently and the ability to proactively adapt to changes in society, in other words, the "power for living." Naturally, guidance for such a learning approach should be thought of in terms of not only textbooks but also the entire school spectrum and activities. Here, we examine the ability to learn independently, as fostered through the study of elementary school mathematics in particular.

The development of children who learn independently requires, first of all, teachers who are sufficiently aware of the importance of such endeavor and rigorous consideration of the type of guidance required to this end.

Furthermore, the ideal form of mathematics classes to teach fundamentals and basics to pupils has been considered based on the above. Fundamentals and basics are not just achieved through a one-hour-a-week class but need to be acquired over many hours (throughout the school year or throughout the course unit). The study of elementary school mathematics is characterized by a spiral-shaped progression of various component areas, with each area having a distinct study style, and it is necessary to promote practical research in the aspects of fundamentals and basics as well as the ideal guidance approach for each area, the problem-solving process per unit time, and the interrelation and realization of learning fundamentals and basics. In this paper, the author considers an approach to mathematics particularly from the aspect of fundamentals and basics.

2. Approach to Mathematics from the Aspect of Fundamentals and Basics

The aims of elementary school mathematics in Japan are the acquisition of fundamental and basic knowledge and skills regarding quantities and geometric figures and, based on this, the cultivation of the basics of creativity—such as the ability to think about things from a number of different aspects and the ability to think logically—as well as the development of an understanding of the merits of investigating and handling phenomena mathematically and the attitude of applying the knowledge thus obtained to subsequent objects of study or daily life. These aims include the definition of the type of mathematics instruction to be realized in the classroom. In elementary school mathematics, the cultivation of the ability to solve problems has been promoted for the formation of "how to learn." For the type of problem solving aimed for in elementary school mathematics, what is expected is the formation of the attitude of developing, through the solving of problems, new ways of looking and thinking about things that can be shared with others in the class and, in this manner, improve one another and achieve self-growth. The formation of how to learn demanded by the Ministry of Education, Culture, Sports, Science and Technology has been achieved based on the school research theme of developing the ability to solve problems. In this sense, in Japan's elementary school mathematics, developing the ability to learn and developing the ability to solve problems are considered to be almost synonymous.

Normally, when learning problem-solving skills, pupils progress through five stages. From the viewpoint of the formation of how to learn, the fundamentals and basics to be taught at each stage are as follows:

(1) Problem setting
Stage at which problems for study are created through the introduction of course units, class periods, etc.
- Ability to set learning targets and set learning sequences
- Ability to grasp the aims of the teacher's lesson progression
- Acquisition of the mathematical way of thinking

(2) Problem solving
Stage at which pupils grasp the situation and understand what the problem is
- Ability to form one's own question from the problem (visualization of situation and formulation of questions)
- Ability to share information about the problem with classmates
- Ability to engage in analogical reasoning from previous learning

(3) Solution planning
Stage at which pupils determine the direction required for solving a problem that has been understood
- Ability to recall previous learning contents and experience (thinking in terms of units, etc.)
- Ability to make predictions (prediction of meaning of calculation and estimation of solution)
- Ability to attempt problem solving on one's own (reexamination of calculation method)
- Intuition ability

(4) Solution execution
Stage at which a solution to a problem is attempted based on the solution plan and a tentative conclusion is drawn
- Ability to execute a solution based on a plan
- Ability to utilize past learning experience and contents (deduction from past learning)
- Ability to express one's thoughts in a manner understandable to classmates (actions

of thought,
judgment, and expression)

- Ability to recognize differences between one's own and one's classmates' thinking and to clarify the essence forming the background of these differences
- Ability to look back on one's actions (functional solution)

(5) Solution study (including review)
Stage at which pupils adequately evaluate processes and achievements and clarify what they have understood and what should be further investigated

- Ability to sympathize with the views of others, going beyond differences in thinking
- Ability to study solutions and come up with better results (perception of integration and development, refining of solution, integration with one's past learning, and acquisition of knowledge and skills)
- Ability to make generalizations and developments (generalization of a line of thought, abstraction, and logical processing)
- Ability to compare what has been learned in class with one's own and one's classmates' thinking and to evaluate oneself (validity of estimation)

The fundamentals and basics at these stages involve many different aspects, such as interest-, motivation-, and attitude-related aspects; mathematical thinking-related aspects; thinking ability-; judgment ability-; and expression power-related aspects; and knowledge- and skill-related aspects.

This section describes the mathematical way of thinking. The mathematical way of thinking is considered to form the core of mathematical education and the base from which arithmetic and mathematics knowledge and skills are produced. Let us examine the approaches discussed below, focusing on school year development equivalents in relation to children's awareness of the benefits and their application thereof through the learning contents in each area.

(1) Mathematical approach related to content—concept formulation, principles and rules
(2) Mathematical approach related to method—thinking in a mathematical manner
(3) Mathematical approach as a logical way of thinking—thinking in a logical manner
(4) Mathematical approach as an integrated and developmental way of thinking—thinking in an
integrated and developmental manner
(5) Mathematical approach that promotes a mathematical sense and thinking ability—numerical sense, quantitative sense and graphic sense
(6) Mathematical approach that promotes a mathematical attitude—approach to problem solving

・ Concept formation, principles and rules: Thinking in terms of units, thinking in terms of place value numeration, thinking in terms of correspondence, percentages, etc.
・ Thinking mathematically: Idealization, encoding, simplification, formalization, compaction, etc.
・ Thinking logically: Analogic thought, inductive thought, deductive thought, etc.
・ Thinking in an integrated and developmental manner: Abstraction, generalization, expansion, etc.
・ Sensory and thinking ability: Estimation, sense of volume, approximation, etc.
・ Arithmetic and mathematical attitude: Utilization of previous learning, outlook, perception of value, etc.

Interest, motivation, and attitude stimulate the intellectual curiosity of the child and, as such, are important motivating forces in the study of mathematics. Each one of these is considered a mental disposition that is actively expressed on one's own from the viewpoints of "the mathematical way of thinking and expression, processing, knowledge and understanding" and

"ways of learning and problem-solving skills." These are positioned as "items that support the fundamentals and basics of content and method." The learning process of pursuing the value and significance of mathematical value is necessary in developing mental disposition.
In the following section, we will focus more particularly on fundamentals and basics related to introduce practices.

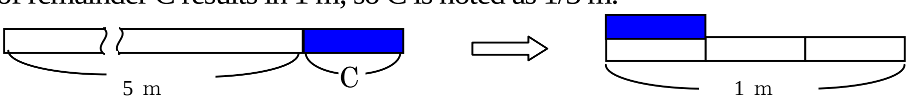
3. Formation of the mathematical way of thinking related to content: The case of thinking in terms of units

With regard to the formation of the mathematical way of thinking discussed in this paper, let us examine the way of thinking about units, from the aspect of content in particular. Let us consider the concept of units when studying the concept of numbers in the area of numbers and arithmetic (integers, decimals and fractions) and units during the numerical conversion of amounts related thereto.

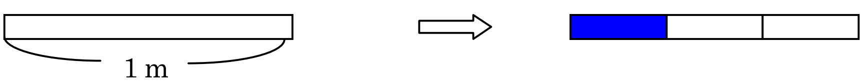
1) **Formation of the concept of units during the number concept formation process**

The acquisition of the number concept begins in the first year, and by looking at things from the same viewpoint, through activities in which one group is created and through one-on-one correspondence between two or more groups, pupils learn numeric representation and relations of magnitude among numbers through division into classes among these groups and through the number of class factors. Moreover, in learning about amounts and estimation—which are closely related, as will be noted later—first-year pupils form the foundation of later numerical linear algebra, as expressed by "length equivalent to x number of erasers," to take length as an example. Here, one piece is used as the unit. In their second year, pupils learn multiplication. In the case of "4 times 3," 3 is seen as a unit of addition repeated 4 times. In this case, the fact that numbers 1 through 9 are each units that can be counted is used. Such activities are the basis of pupils' understanding of the concepts of decimals and fractions over four years of instruction.
Decimals are introduced in the fourth year as numbers used in expressing the fractional part in relation to the estimation of amounts. For example, when performing a measurement using a 1-m ruler, when the measured length is "1 m plus," based on the concept of decimal notation, that "plus" fraction is processed with the concept of dividing the unit (1 m) into 10 equal parts and then using one of these 10 subdivisions as the new unit, the pupil learns numerical conversion of fractions into a number of such new units based on this.
Moreover, when measuring amounts using unit quantities—unlike in the case of decimals, where the decimal notation system is the basis for expressing fractional amounts—the introduction of new numbers and fractions comes to mind. The following two methods come to mind as such a method. Let us consider, for example, the case of 1/3.

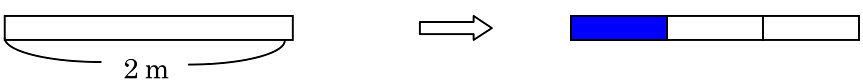
□ When an object is measured using 1 m as the unit, remainder C remains. Combining three of remainder C results in 1 m, so C is noted as 1/3 m.



□ Let us divide 1 m by 3 to obtain 1/3 m.



□ and □ are the same amount but are different ways of thinking. This difference is clearly shown in □' below.
□' 2 m is divided by 3, and 1 segment is denoted as 1/3 m (incorrect).



Both □ and □ can be called fractions that express fractional quantities in the sense that they

express an amount, but □ is an amount representation in the extension of fractions of the part-whole by equal splitting (several equivalent parts obtained by splitting the whole into equal parts) and leads easily to such a concept as that in □’. In the case of fractions of the part-whole by equal splitting, the whole is regarded as 1, and the action of splitting this whole into equal parts is emphasized, whereas when obtaining the number of split parts by dividing the unit amount of 1 m by the remainder amount and expressing the size of one such part, the remainder amount becomes the unit. In approach □, the approach of a given number of the unit fraction becomes clear. The approach of expressing 1 m with the remainder in □ is demonstrated by the Euclidean algorithm. Actually, this approach lies in obtaining the greatest common divisor c in relation to two natural numbers a and b[1].
The arithmetic action of using the remainder that remains following division as a unit that is again used for measurements is clearly different from the action of splitting the unit in □ into equal parts and using the expression as several of these parts. Moreover, it is important to consider such decimals and fractions as 0.2 and 2/7 in terms of two new units of 0.1 and 1/7, respectively. In this manner, the approach of thinking in terms of a new unit and creating such a unit plays a very important role when thinking about the four arithmetical operations for decimals and fractions. In other words, let us consider the sum of decimals and fractions as follows.

Content of “Amounts and Measurements” That Are Being Taught

In Japan, pupils learn the numerical conversion of amounts using units when being taught “amounts and measurements,” broken down by school year, as follows. Such a structure aims to have pupils learn about the existence of various amounts and corresponding units as well as develop an awareness of units.

First year: Lengths (direct comparison, indirect comparison and measurements using arbitrary units)

Second year: Lengths (meaning of units and measurements and measurements using universal units (cm, mm and m))

Third year: Length (ability to make estimates and measurements using universal units (km))

Bulk and volume (concepts and measurements using universal units (l, dl and ml))

Time of day and time (duration) (concept, units (day, hour, minute and second), obtaining the time of day and time (duration))

Fourth year: Extent and area (concept, area of squares and rectangles, measurements using universal units (cm^2, m^2 and km^2) Angle size (concept and measurements using a universal unit (degree (°))

Fifth year: Quadrature formula for areas (areas of a triangle, parallelogram and circle)

Sixth year: Bulk and volume (concepts of volume, cubes, rectangular parallelepiped and measurements using universal units (cm^3, m^3)

Regarding the various types of amounts, in the measurement stage, pupils learn numeric conversion into a number of that given unit (arbitrary unit or universal unit). However, to make pupils understand the meaning of such measurements, it is important to make them practice repeatedly, using arbitrary units. Through the repeated implementation of such instruction, pupils deepen their understanding of units.

1. The Euclidean algorithm is a method of certifying the mathematical world by performing measurements with units. Assuming natural numbers a and b, with a > b, we obtain $a \div b = q_1 + r_1$ ($0 □ r_1 < b$). If $r_1=0$, we obtain the greatest common denominator b. If $r_1 \neq 0$, $b \div r_1 = q_2 + r_2$ ($0 □ r_2 < r_1$), and if $r_2=0$, we obtain the greatest common denominator of a, b and r_1. If $r_2 \neq 0$, the same operation is repeated, so that when $r_1 \div r_2 = q_3 + r_3$ ($0 □ r_3 < r_2$), ……, $r_n \div r_{n+1} = q_{n+2}$ results, the greatest common denominator of a and b is r_{n+1}.

2) Case: Formation of the mathematical way of thinking in courses introducing decimals

The introduction of decimals is discussed using cases (see appendix). The instructors are trainee teachers (third-year students), yet they can be called good instructors even when compared to currently active teachers. The inculcation of many different ways of thinking is sought in one class. In this class, the instructor aims to form the following types of mathematical thinking, including the concept of units.

 Understanding the concept of decimals (Concept formation: Concept of decimals)

By the time they start the course, pupils have learned the concept of integers greater than 0, the meaning of the four arithmetic operations and how to perform calculations. During the course, through the use of fluid volumes, pupils learn decimal notation using new units for amounts that cannot be expressed with integers using l as a unit. The point at which ingenuity is used in the course is when instructors have pupils perform the numerical conversion of amounts while keeping them interested by preparing the same amounts of liquid in different containers by group and using water of different colors. The pupils are made to write the amounts using cups listed in the worksheets provided by the instructor and are taught in an easy manner through activities in which they perform numerical conversion of odd amounts.

Using the fact that 1 dl, a unit the pupils have learned by then, is one tenth of 1 l, pupils are shown that this tenth is expressed as the decimal 0.1 in relation to the original 1 l unit, and they learn that 2 dl can be represented as 0.2 l.

 Forming an attitude that facilitates making estimates (Sense and thinking ability: Estimates)

One more important thing about this course is that, after providing cups containing colored water, the teacher should ask the pupils, “How much water do you think they contain?” When learning about arithmetic and mathematics, it is important to acquire a sense of quantity.

 Expressing amounts with 1 unit (Thinking mathematically: Simplification and integration)

From what they have learned up to this point, pupils have acquired the understanding that the amount obtained is “1 l and 2 dl,” but they must learn how to express this amount more simply using just one unit: l. In learning arithmetic and mathematics, pupils must understand that the simplest expression is desirable.

4. Mathematical Tools That Must Be Provided to Children

The term *mathematics class* may be misconstrued as practicing calculations. In a class where one learns and thinks for oneself, the mathematical way of thinking with regard to method and the mathematical way of thinking with regard to content are simultaneously acquired by the child. The implementation of problem-solving classes consists in allocating various problem-solving methods throughout the entire class, making pupils learn and accept each other's way of thinking, thereby teaching them the benefits of thinking. During this process, children learn how to think in relation to content and method from having to find ways to express their thoughts to each other.

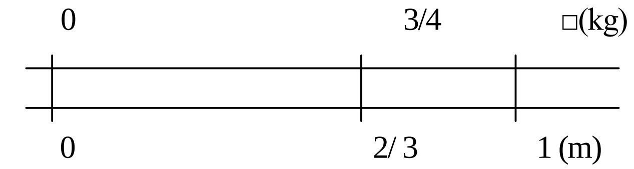
In order for children to become able to skillfully express content and method, they need to acquire the tools they will need when thinking about ways to express these things. Actually, as thoughts are exchanged, even when, for example, one feels that a tool (method of expression) or way of thinking used by a child in class is effective, there will likely be few opportunities to re-present situations so that all the children will be able to use such tools.

In the process of learning, the presentation of opportunities for the child to choose his/her own useful tools when thinking mathematically is important for forming the mathematical way of thinking. In so doing, it is also important to enable the child to use the same tools continuously and to develop these tools to create new ones. The term *problem solving* involves an emphasis on opportunities to make discoveries, but such discoveries, rather than being sudden occurrences, are for the most part achieved based on previous learning. If anything, it is important to value learning opportunities regarding ways to use tools as a part of the thinking process and ensure that children have an ample supply of such needed tools as they tackle problem solving.

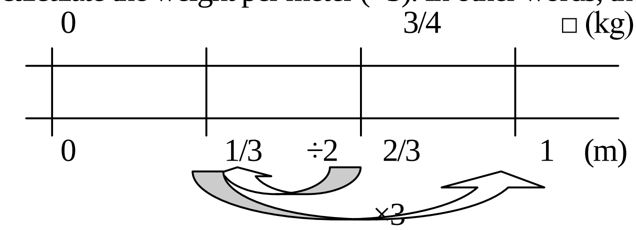
For example, in the area of numbers in arithmetic, children can use line charts or surface diagrams to show relationships between quantities and explain what they are thinking. Even though they know that these line charts and surface diagrams are effective, there are probably many children who do not understand how to use these tools. In this sense, it is important to teach them how to use line charts and surface diagrams as tools for thinking. In the following, let us take up the case of two number lines as an extension of line charts.
Example: Let us consider the following problem.

If an iron rod measures 2/3 m and weighs 3/4 kg, how many kg would it weigh if it were 1 m?

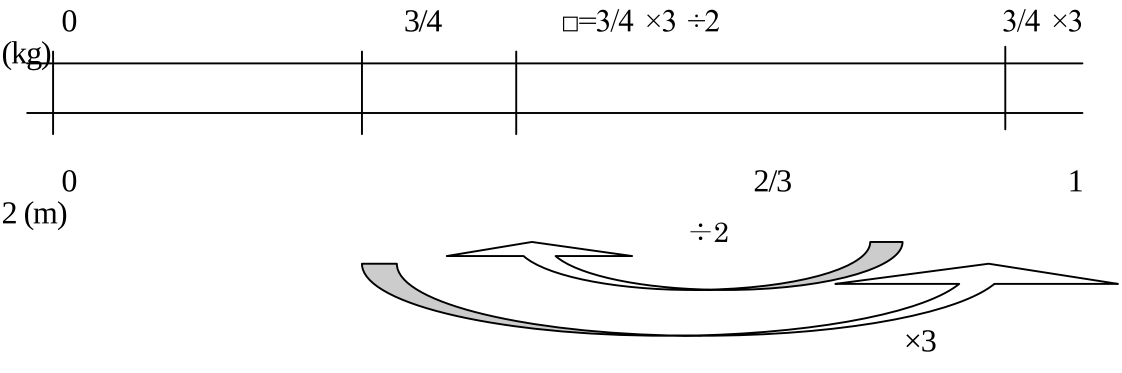
The following diagram can be used to understand that the formula for solving this problem is 3/4 ÷ 2/3 and that this is the means of solving this problem. (Because we used division to obtain the amount per meter in the integer problem, we shall use division here too.)



Using this diagram, let us think in the following way, for example, to obtain the weight of a 1-m iron rod.
□ To obtain the weight per meter, first let us obtain the weight of 1/3 m (÷2) and then calculate the weight per meter (×3). In other words, the formula is 3/4 ÷ 2/3 = 3/4 ÷ 2 × 3.



□ To obtain the weight per meter, first let us obtain the weight of 2 m（×3）and then calculate the weight per meter (÷2). In other words, 3/4 ÷ 2/3 = 3/4 × 3 ÷ 2.



In this manner, in classes that value children solving problems on their own, it is necessary that pupils learn how to use the tools that enable them to think for themselves. The ability to use the two number lines is acquired, by its very nature, through many hours of practice, and children that do not know how to use it at first gradually learn. Also, the two number lines require five or six years to master and are not something in which one can excel overnight. Textbooks are devised so that the line charts to be used by first and second year pupils are systematically and continuously addressed and are learned as an extension of line diagrams.

References:

Shigeo Katagiri, New Edition, “Mathematical Thinking and How to Teach It (I),” *Mathematical Thinking and How to Teach It*, Meiji Tosho (2004)

Shigeo Katagiri, “Mathematical Thinking and How to Teach It,” *Progress report of the APEC Project: “Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (□),”* University of Tsukuba (2007) pp. 105–158.

Kazuyoshi Okubo, “Mathematical Thinking from the Perspectives of Problem Solving and Area Learning Contents,” *Progress report of the APEC Project: “Collaborative Studies on Innovations for Teaching and Learning Mathematics in Different Cultures (□),”* University of Tsukuba (2007) pp. 237–244.

Appendix

Fourth Year Elementary School Mathematics Teaching Plan

Date & Time: September 18 (Thu.), 2003; 5th period
Pupils: Fourth-Year Class 3; 15 boys, 15 girls; total: 30
Instructor: Meien Elementary School , Sapporo City(Education Trainee) Fumito Chiba

1. Unit Name: Decimals
2. Unit Aims

To teach pupils the use of decimals to express the size of fractional parts that do not reach the unit quantity and have them use decimals appropriately

- Have pupils realize that the size of fractional parts that do not reach the unit quantity is expressed with decimals and have them readily try to use decimals.
- Have pupils realize that decimals, like integers, are expressed using the decimal notation system.
- Have pupils realize that, based on the decimal system, addition and subtraction operations can be done in the same manner as for integers by calculating numbers of the same place value.
- Teach pupils how to express the size of fractional parts that do not reach the unit quantity using decimals.
- Teach pupils how to view, in a relative way, numbers expressed with decimals based on 0.1, etc.
- Teach pupils how to express decimals on a number line and read decimals displayed on a number line.
- Teach pupils how to add and subtract decimals in simple cases.
- Have pupils understand the meaning of decimals and how to represent them.
- Have pupils understand decimal addition and subtraction.

3. Regarding Course Units

Up until this point, what the pupils have learned about amounts consisted of clarifying units and learning that the number of such units that can be expressed with an integer. Here, however, pupils will learn to estimate the size of amounts smaller than units, i.e., fractions, and how to express these.

To express the size of a quantity that is smaller than a unit, one uses decimals and fractions. Here, however, the division number differs from arbitrary fractions, with the unit being divided into 10 equal parts, and decimals that can be combined in groups of 10 for a decimal scaling position (decimal system) are taken up.

Fourth-year pupils deal with decimals through three course studies, namely, (1) the expression of fraction size, (2) the decimal system and (3) the calculation of decimal addition and subtraction on paper. This class period will introduce decimals as described in (1) and will cover amounts measured in liters. By using a new unit created by dividing 1 l, which is the unit quantity, into 10 equal parts, we will have pupils work on the size of fractions.

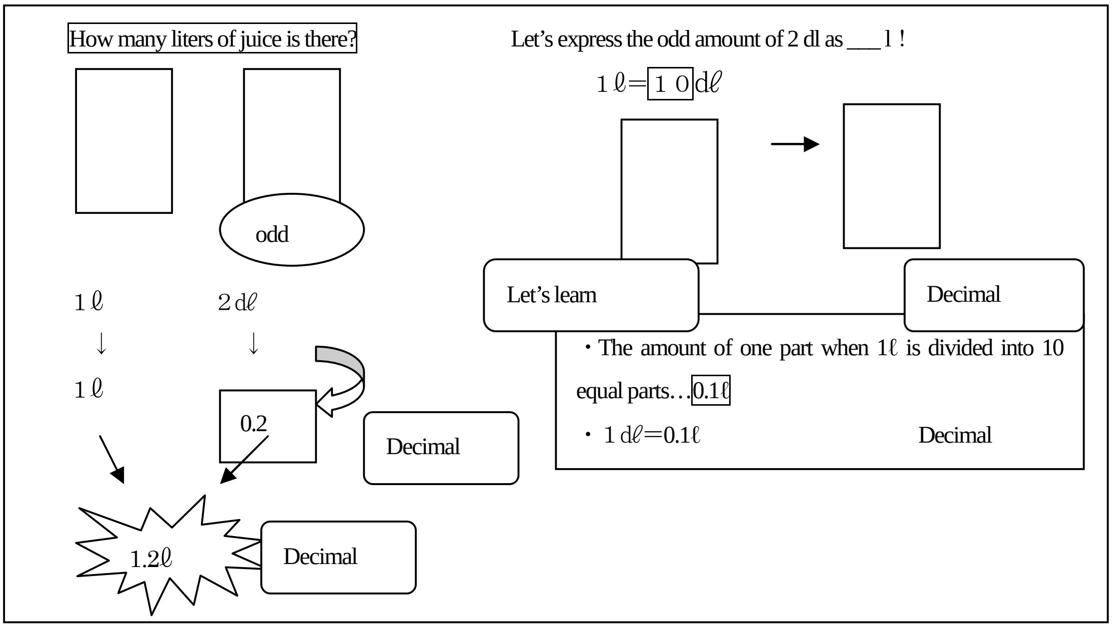
In this class period, by having children engage in various activities, such as using colored water to transfer an amount that is less than 1 l into a graduated container by hand while visually checking the operation, we will help them understand that odd amounts can be expressed with decimals.

Pupil Activities	Role of the Teacher

4. Teaching Plan (8 Hours)

Subunit	Duration (hrs.)		Teaching Contents
1. Expression of fraction size	2	1	Using decimals to express the size of a fraction that does not reach the unit quantity
		1	The ability to express the size of fractions using decimals, even in the case of length (cm)
2. Decimal system	3	1	・Number line display of decimals ・Meaning of the term *one decimal place* and the decimal scaling position of decimals
		1	Relative size of decimals and structure and magnitude of numbers
		1	Addition and subtraction of decimals in simple cases
3. Addition and subtraction of decimals	2	1	Addition of decimals by hand up to one decimal place
		1	Subtraction of decimals up to one decimal place
4. Practice and review	1		Review of this class period and practice

5. Blackboard Plan



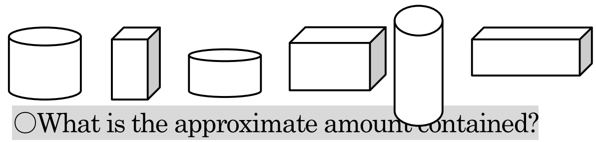
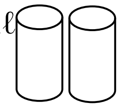
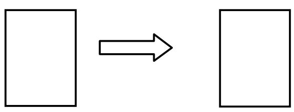
(2) Expansion of this class period (1/8 hr)

6. Lesson of This Class

(1) Course aim

Understand that decimals are used to measure amounts less than the unit quantity.

(2) Course development (1/8 of the class time)

 ○What is the approximate amount contained?	・Beforehand, arrange the desks to make six groups Have the children estimate the amount. ・Explain the measurement method (1 l container without graduations) ・Walk around the desks and check how the children are doing.
1ℓ  Odd amount○How much is the odd amount? About 2 dℓ Less than half of 1ℓ 3 dℓ !	Note: Ensure that the children work carefully without spilling anything. ・Define the term *odd amount*: This is an amount that is less than 1 l (unit quantity).
Estimate: Total amount of juice is 1 ℓ and ○dℓ. ○What would you need to know the exact amount?	・Have the children estimate odd amounts using dl, which they learned in their 3rd year.
Let's use a graduated container to obtain the amount. Total juice is 1 **ℓ and** 2 dℓ!! How many □**ℓ** of juice is there in all? Pupils realize that 2 dℓ (odd amount) can be expressed in ℓ.	・Distribute graduated containers. ・Explain that regardless of their different shapes, each container holds the same amount. ・Talk about the benefit of converting 1 l and 2 dl into ___ l. ・Distribute printouts. Have the pupils review what they have learned up to that point.
Let's express how many l 2 dl (odd amount) are. ○Let's accurately write 2 dl in the left box on the printout.  Pupils realize that, similar to dl, l can be divided into 10 equal parts.	・Confirm that 1 l = 10 dl. ・Walk around the desks and check how the children are doing. ・For children who do not understand the concept of 10 equal parts, suggest that the height of the box is 10 cm.
Pupils learn that the amount corresponding to one of the 10 equal parts that 1 l has been divided into is 0.1 l (1 dl = 0.1 l). Amount of odd amount: 2 dℓ=**0.2ℓ** ! ! ○How much is 1 l plus 0.2 l?	・Explain using the figures (boxes). Do not make detailed reference to addition.

1.2ℓ of juice in all Review The size of the odd amount can be expressed with decimals. Input of impressions, note regarding the next class period	